Message Lengths for Noisy Network Coding

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Distinguished UMIC Lecture
RWTH Aachen, Feb. 13, 2012
Outline

1) What is Cooperative Communications?
2) What is Network Coding?
3) Relaying via Noisy Network Coding
1) What is Cooperative Communications?

Network communication where nodes cooperate, rather than compete, to transmit data for themselves and others

- Classic networks: TDM/FDM, admission control, routing
- Question: how should devices best operate?
  To answer this question fundamentally we need …
Network Information Theory

- Two Pioneers: Ahlswede and El Gamal
- Two of El Gamal’s many important contributions:
  1) Cut Bounds for information networks (1978)

R. Ahlswede (15.9.38 – 18.12.10)  A. El Gamal

2006 Shannon Lecturer  2012 Shannon Lecturer
Two-Way Channel (Capacity an Open Problem)

\[ R_1 = \frac{B_1}{n} \text{ bits/use} \]
\[ R_2 = \frac{B_2}{n} \text{ bits/use} \]

- Shannon’s Capacity Bound: given \( P(x_1, x_2) \) we have
  \[ R_1 \leq I(X_1; Y_2 | X_2) \quad R_2 \leq I(X_2; Y_1 | X_1) \]

- El Gamal’s Cut Bound: partition network nodes into 2 sets and develop similar bound. Method applies to any information network (biological, physical, financial, social, etc.)
2) What is Network Coding?

- Consider a classic network. For each edge \((i,j)\), choose \(f_{i,j}(.)\) to “uniformly” map \(\{y_i\}\) to \(x_{i,j}\)
- Linear coding: \(x_{i,j} = A_{i,j} y_i\) where \(A_{i,j}\) is often taken to be random

Ahlswede-Cai-Li-Yeung (2000)
Example: Traffic Network

- Consider a traffic network with capacities in cars/minute. How many **cars** can flow between nodes 1 and 2 per minute?
- The bottleneck is clearly street (3,4). The answer is 10 cars per minute, either red or blue.
- But the answer is **different** for **digital communication** networks.
Example: Communication Network

- Bottleneck: 10 Mbit/sec (say) but now both nodes can send 10 Mbit/sec simultaneously by using network coding
- Trick: node 3 takes bits $B_1$ and $B_2$ from nodes 1 and 2, respectively, and sends bit $C = B_1 \oplus B_2$ to node 4
- Node 1 computes $B_2 = C \oplus B_1$ and Node 2 computes $B_1 = C \oplus B_2$
- Many beautiful recent (2000-) results using Galois field algebra
Network Coding for Wireless

- Nodes have interference and broadcast constraints
  For each node $i$, choose $f_i(.)$ to map $y_i$ to an $x_i$ (but how?)
- Non-linear $f_i(.)$ needed in general
- $x_i$ are independent if one uses “independent uniform” maps

![Diagram of network coding](image.png)
Example: Satellite Network

- Use “decode-forward”; node 3 broadcasts $C = B_1 \oplus B_2$
- Savings: $\leq \frac{3}{4}$ freq/time or large energy gains via coding
- Demonstrator: TUM-LNT/DLR-IKN/IQW collaboration
3) Relaying
The core of cooperative communications is relaying
- Above: network coding, decode-forward
- Question: are there other good strategies?
  To answer this question fundamentally we first study a basic ...
Relay Channel (Capacity an Open Problem)

- Problem: maximize $R$ for reliable communications
  ($B$ and $n$ are permitted to be large)

$X_1 \rightarrow P(y_2, y_3 | x_1, x_2) \rightarrow X_2 \rightarrow Y_2 \rightarrow Y_3 \rightarrow M$

$B$ message bits
$n$ channel uses
$R = B/n$ bits/use
Known Relaying Strategies

Basic Methods
1) Amplify-Forward (AF): amplify $Y_2$
   Memoryless Relaying: forward $f(Y_2)$ with optimized $f(.)$
2) Decode-Forward (DF): decode message and re-encode

Recent for Large Networks: Compute-Forward via Lattice Coding

Compression-Based Methods
1) Classic Compress-Forward (CF), 1979
2) Quantize-Map-and-Forward (QF), 2007
3) Noisy Network Coding (NNC), 2010
4) Short-Message NNC (SNNC), 2010
   (aka "Cumulative encoding/block-by-block backward decoding")
Structure of Compression Methods

- **Two-step**: (1) compress (quantize/hash) and (2) channel code
- Method is **digital** (binary interface) and **non-linear** (in general)
- Mapping is “independent and uniform” across interface
- Surprise: Includes classic network coding as a special case!
1) Classic QF

- Relay quantizes $Y_2$ to bits $q$ representing $\hat{Y}_2$ and transmits $x_2(q)$
- Simple: use scalar quantization (good for high-rate quantization)
  Better: use vector quantization with distortion $D$ after canceling effect of $X_2$: $I(Y_2; \hat{Y}_2 | X_2) < R_Q(D)$
- Reliable transmission rate: $R_Q(D) < I(X_2; Y_3)$
**Classic CF**

- **Improvement #1**: relay hashes $q$ (aka Wyner-Ziv coding)
  Quantization bound improves to: $I(Y_2; \hat{Y}_2|X_2Y_3) < R_Q(D)$

- **Improvement #2**: bursty transmission helps at low SNR, i.e.,
  use high power for short time intervals. Formally take into account via a “time-sharing” random variable $T$. 

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<table>
<thead>
<tr>
<th>Source</th>
<th>Block 1</th>
<th>Block 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{11}(m_1)$</td>
<td>$x_{12}(m_2)$</td>
<td>$x_{22}(h(q))$</td>
</tr>
<tr>
<td>$0$</td>
<td>$x_{22}(h(q))$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\hat{y}_{21}(q)$</td>
<td>$\hat{y}_{21}(q)$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
CF Rate

- **Final CF Rate**: with a cut-set interpretation

\[ R < \max \min \left[ I(X_1; \hat{Y}_2 Y_3 | X_2 T), I(X_1 X_2; Y_3 | T) - I(Y_2; \hat{Y}_2 | X_1 X_2 Y_3 T) \right] \]

2),3) QF/NNC*: Network Coding for Wireless and Beyond

- Source repetitively encodes a long message $m$
- Relay quantizes only (no hashing)
- Destination decodes $m$ and $q$ jointly
- Advantage: theory extends nicely to many sources and relays
- Issues: long (en/de)coding delay, limited DF possibilities

* Avestimehr et al. (Allerton 2007), Lim et al. (ITW 2010)
4) SNNC*

- **Improvement**: classic short messages \(m_1,m_2\) achieve the same rates (proof below); can use **backward** decoding
- **Minor generalization**: use long message \(m\) but hash \(m\) to short messages \(h_1(m)\) and \(h_2(m)\)
- **Advantage**: enables DF to improve reliability for slow fading

\[
\begin{array}{c|c|c}
\text{Source} & \text{Block 1} & \text{Block 2} \\
\hline
x_{11}(m_1) & 0 & x_{12}(m_2) \\
\hline
\hat{y}_{21}(q) & 0 & x_{22}(q)
\end{array}
\]

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* Wu-Xie (2010), **Kramer-Hou (2011)
Experiment

- Single-relay, ½ way between source and destination
- Attenuation exponent 3, slow Rayleigh fading, Gaussian noise
- Per-node power constraint
- Rate = 1 bit/s/Hz
- SNNC gains 2 dB over QF/NNC at outage prob. $10^{-3}$
- Gains reduce for higher rates
Proof of Equivalence for 1 Relay

- Consider a fixed coding distribution
- QF/NNC rate with long messages and joint decoding:
  \[ R < \max\{ I(X_1; Y_3), \min [ I(X_1; \hat{Y}_2 Y_3|X_2 T), I(X_1X_2; Y_3|T) - I(Y_2; \hat{Y}_2|X_1 X_2 Y_3 T) ] \} \]  (1)
- Additional bound for backward decoding:
  \[ 0 \leq I(X_2; Y_3|X_1 T) - I(Y_2; \hat{Y}_2|X_1 X_2 Y_3 T) \]  (2)
- Suppose (2) is violated. Subtract right-hand side of (2) from the 3rd expression in (1) to get
  \[ R < \max\{ I(X_1; Y_3), I(X_1; Y_3|T) \} = I(X_1; Y_3) \]
- Proof method generalizes to many relays and sources

* Destination treats \( X_2 \) as noise
Discussion: Deterministic and Gaussian Channels

- R < max min [ I(X_1; \hat{Y}_2 Y_3|X_2 T), I(X_1X_2; Y_3|T) - I(Y_2; \hat{Y}_2|X_1X_2Y_3T) ]
- Deterministic channels have Y_2=f(X_1,X_2) so choose \hat{Y}_2=Y_2 and achieve a cut-bound with independent inputs
  (Note: capacity known and achieved by “Partial DF”)

- Gaussian channel: choose \hat{Y}_2=Y_2+\hat{Z}_2 where \hat{Z}_2 \sim N(0,N_2). Get
  - I(Y_2; \hat{Y}_2|X_1X_2Y_3T=1) = I(Z_2; Z_2+\hat{Z}_2) = \log(2N_2/N_2) = 1 bit
  - I(X_1; Y_2 Y_3|X_2 T=1) - I(X_1; \hat{Y}_2 Y_3|X_2 T=1) \leq \log(2) = 1 bit

- R is within 1 bit of the cut-set bound with indep. X_1 and X_2
  (High SNR: beamforming gains are small so virtually optimal)
- Low SNR: bursty signaling useful
Many Nodes, either Sources or Relays

- QF/NNC properly extends classic network coding*
- **SNNC:**
  - get same rates for 1 source and many relays (Wu-Xie 2010)
  - get *same* rates for many sources and relays (K-Hou 2011)
  - facilitates DF (K-Hou 2011)
  - joint decoding avoids block-Markov coding at sources for DF**

* Lim et al. (ITW 2010), ** Hou (2011)
Proof of Equivalence for 1 Source*

- Fix a coding distribution. Let $V$ be the set of relays. Let $S \subseteq T \subseteq V$ and $\hat{S}$ be the complement of $S$ in $T$. Define

$$R_T(S) = I(X_1 X_S; Y_{\hat{S}} | X_{\hat{S}}) - I(Y_S; \hat{Y}_S | X_1 X_T Y_{\hat{S}})$$

$$Q_T(S) = I(X_S; Y_{\hat{S}} | X_1 X_{\hat{S}}) - I(Y_S; \hat{Y}_S | X_1 X_T Y_{\hat{S}})$$

- QF/NNC bounds: $R \leq \max_T \min_S R_T(S)$ \hspace{1cm} (1)

- **Backward** decoding: $T$ must satisfy $0 \leq Q_T(S)$ for all $S \subseteq T$ \hspace{1cm} (2)

- Suppose (2) is violated for some $S$. Then for all $B$ with $S \subseteq B \subseteq T$ we have $R \leq R_T(B) < R_T(B) - Q_T(S) = R_{T \setminus S}(B \setminus S)$

- This means the destination can treat the $X_k$ with $k \in S$ as noise

- Repeat argument until all bounds (2) satisfied

- Proof method generalizes to many sources (ISIT 2012, submit.)

* Kramer-Hou (ITW 2011)
Discussion

- $R_S < \max \min_{(S,\hat{S})} I(X_S; \hat{Y}_S Y_d | X_S T) - I(Y_S; \hat{Y}_S X_S X_S Y_S Y_d | T)$
- **Deterministic** (e.g. classic) networks: choose $\hat{Y}_i = Y_i$ and achieve cut-set bound with **independent** inputs
- **Gaussian** networks: choose $\hat{Y}_k = Y_k + \hat{Z}_k$, $\hat{Z}_k \sim \text{CN}(0, N_k)$, to get within $0.63|V|$ bits of the cut-set bound (here a **true** upper bound with **dependent** inputs)
- Problems inherent to long messages:
  - Encoding and decoding **delays** are large (latter problem also for joint or backward decoding)
  - Must hash $m$ for reasonable modulation set sizes
  - **Inflexible**: relays cannot use multihop/DF
Application Question

Does QF/NNC have a practical future?

- relays can operate in a **distributed and autonomous** fashion
- achieves the “multi-output” gains of MIMO
- SQF/SNNC with DF achieves “multi-input” gains of MIMO
- method applies to more than radio, e.g., classic & optical networks

**Difficulty and Research:** how to design practical codes and decoders?
Research Activities at LNT

- Timing Codes
- Low-Energy Cooperation
- Interference Management
- Fiber Optic Capacity
- Low Complexity Decoders
- Satellite Network Coding
- Distributed Storage Coding
- Network Erasure Coding
- Information & Coding & Communications Theory

GOOD AND GREEN RADIO
Extra Slides
Gaussian Relay Channel

- Gaussian noise $Z_t$ (variance $N$), $t=2,3$
- Cost: $\sum_i |X_{ti}|^2/n \leq P_t$, $t=1,2$
  (or use total power, peak power, etc.)